Math 3280 Tutorial 6.

Recall: Probability density function for a continuous r.v. X

$f: \mathbb{R} \to [0,\infty)$

$$P(X\in A) = \int_A f_X(x)\,dx. \quad A\subseteq \mathbb{R}.$$

- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$
- $F(b) := P(X\le b) = \int_{-\infty}^{b} f_X(x)\,dx.$

  F: cumulative distribution function
- $E[X] = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx$, $\quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\cdot f_X(x)\,dx.$

② Uniformly distribution on $[a,b]$:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x\in[a,b] \\ 0 & \text{otherwise} \end{cases}$$

$E[X] = \frac{a+b}{2}$, $\quad Var(X) = \frac{(a-b)^2}{12}.$

③ normal r.v. with parameters (μ,σ^2):

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x\in(-\infty,+\infty).$$

$E[X] = \mu$, $\quad Var(X) = \sigma^2.$

④ Exponential r.v. with parameter λ:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x\ge 0 \\ 0, & \text{Otherwise}. \end{cases}$$

$E[X] = 1/\lambda$, $\quad Var(X) = 1/\lambda^2$,

$E[X^n] = \frac{n}{\lambda} E[X^{n-1}]$, $n\ge 2.$

Example 1: Let X be a r.v. with density function

$$f_X(x) = \begin{cases} c(1-x^2), & -1 < x < 1 \\ 0 & \text{Otherwise.} \end{cases}$$

① find c.

② Cumulative distribution function $F(x) := P(X \leq x)$.

Solution: ① $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-1}^{1} c(1-x^2)\,dx = 1.$

$$2c - \frac{2}{3}c = 1$$

$$\Rightarrow c = \frac{3}{4}.$$

② When $x \leq -1$, $F(x) = 0$.

when $-1 < x < 1$, $F(x) = \int_{-1}^{x} \frac{3}{4}(1-y^2)\,dy$

$$= \frac{3}{4}\left(x - \frac{x^3}{3}\right) + \frac{1}{2}.$$

When $x \geq 1$, $F(x) = \int_{-1}^{1} f_X(x)\,dx = 1.$

Example 2: Let X be a r.v. with probability density function

$f_X(x) = x e^{-x}$, $x \geq 0$.

Compute $E[X]$.

Solution: $E[X] = \int_0^{\infty} x \cdot x e^{-x}\,dx$

$= \int_0^\infty x^2 e^{-x} dx$

$= \int_0^\infty (-x^2) \, de^{-x}$

<u>integration by parts</u> $e^{-x} \cdot (-x^2) \Big|_0^\infty - \int_0^\infty e^{-x} \cdot d(-x^2)$

$= \int_0^\infty e^{-x} \cdot (2x) \, dx$

$= \int_0^\infty -(2x) \, de^{-x}$

$= (-2x) \cdot e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} \cdot d(-2x)$

$= 2 \cdot \int_0^\infty e^{-x} dx.$

$= 2 \cdot \int_0^\infty (-1) de^{-x}$

$= (-2) \cdot \left(e^{-x} \Big|_0^\infty\right) = 2.$

Exaple 3: For a non-negative r.v. Y, we have

$$\underline{E[Y] = \int_0^\infty P(Y > t) dt.} \quad (1)$$

Use this to prove

$$\underline{E[Y^n]} = \int_0^\infty n t^{n-1} \cdot P(Y > t) dt. \checkmark$$

Solution: $E[Y^n] \stackrel{\text{by (1)}}{=} \int_0^\infty P(Y^n > t)\,dt$

$$= \int_0^\infty P(Y > t^{\frac{1}{n}})\,dt$$

$$\stackrel{y = t^{\frac{1}{n}}}{=} \int_0^\infty n \cdot y^{n-1} P(Y > y) \cdot dy$$

Exaple: 4 let X be a r.v. takes value detwen $0, C$, $P(0 \le X \le C) = 1$.

Show that

$$Var(X) \le \frac{C^2}{4}.$$

Solution: $Var(X) = E[X^2] - (E(X))^2$,

$$\stackrel{(X \le C)}{\le} E[X \cdot C] - (E(X))^2$$

$$= C \cdot E[X] - (E(X))^2 \le \frac{C^2}{4}.$$

$f(x) = Cx - x^2$.

attains maximum at $x = -\frac{C}{-2} = \frac{C}{2}$

$f(\frac{C}{2}) = \frac{C^2}{4}$.

Exaple 5 show that Z is a standard normal r.v.

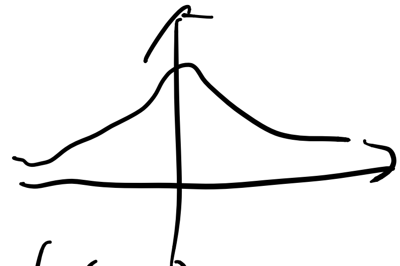
then for $x > 0$,

(a) $P(Z > x) = P(Z < -x)$

(b) $P(|Z|>x) = 2\cdot P(Z>x)$.

(c) $P(|Z|<x) = 2\cdot P(Z<x) - 1$.

Solution: (a). $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$,



$f_Z(z) = f_Z(-z)$.

$$P(Z>x) = \int_x^{\infty} f_Z(z)\,dz = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$\underline{\underline{u=-z}} \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du$$

$$= P(Z<-x).$$

(b). $P(|Z|>x)$

$= P(Z>x) + P(Z<-x)$.

$= 2\cdot P(Z>x)$

(c) $P(|Z|<x) = P(-x<Z<x)$

$= P(Z<x) - \underline{P(Z\le -x)}$

$\underline{\underline{\text{by (a)}}}$ $P(Z<x) - P(Z\ge x)$

$$= P(Z<x) - (1 - P(Z<x))$$

$$= 2 \cdot P(Z<x) - 1.$$

exaple 6) Z be a standard normal r.v.

Show that

(a) $E[Z^{n+1}] = n \cdot E[Z^{n-1}]$

(b) Find $E[Z^4]$.

solution:

(a) $E[n \cdot Z^{n-1}] = \int_{-\infty}^{\infty} n \cdot z^{n-1} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{f_Z(z)} dz$

$$\overset{dz^n = n z^{n-1} dz}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot dz^n$$

$$\overset{\text{integration by parts}}{=} z^n \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} z^n d\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

$$d\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot (-z) dz$$

$$= 0 - 0 - \int_{-\infty}^{\infty} z^n f_Z(z) \cdot (-z) dz$$

$f_Z(z)$,

$$= \int_{-\infty}^{\infty} z^{n+1} f_Z(z)\, dz$$

$$= E[Z^{n+1}].$$

(b). $E[Z^4] \underset{n=3}{\overset{\text{by (a)}}{=\!=\!=}} 3 \cdot E[Z^2] = 3.$

$E[Z^2] = 1$.